Gauge field theories with covariant star-product

JHEP07(2009)084

(http://iopscience.iop.org/1126-6708/2009/07/084)

Published by IOP Publishing for SISSA

Received: *June 8, 2009*
Accepted: *July 2, 2009*
Published: *July 23, 2009*

Gauge field theories with covariant star-product

M. Chaichian,[a] A. Tureanu[a] and G. Zet[b]

[a] *Department of Physics, University of Helsinki and Helsinki Institute of Physics, P.O.Box 64, FIN-00014 Helsinki, Finland*

[b] *Department of Physics, "Gh. Asachi" Technical University, Bd. D. Mangeron 67, 700050 Iasi, Romania*

E-mail: masud.chaichian@helsinki.fi, anca.tureanu@helsinki.fi, gzet@physics.phys.tuiasi.ro

Abstract: A noncommutative gauge theory is developed using a covariant star-product between differential forms defined on a symplectic manifold, considered as the space-time. It is proven that the field strength two-form is gauge covariant and satisfies a deformed Bianchi identity. The noncommutative Yang-Mills action is defined using a gauge covariant metric on the space-time and its gauge invariance is proven up to the second order in the noncommutativity parameter.

Keywords: Non-Commutative Geometry, Gauge Symmetry

Dedicated to Ioan Gottlieb on the occasion of his 80th birthday anniversary.

doi:10.1088/1126-6708/2009/07/084

Contents

1 Introduction

Noncommutative gauge theories have been intensively studied recently (see [1]–[9] for an incomplete list of references). They are defined on noncommutative space-times whose coordinates satisfy the property

$$[x^\mu, x^\nu] = i\theta^{\mu\nu}, \tag{1.1}$$

where in the canonical case $\theta^{\mu\nu}$ is an antisymmetric constant matrix of dimension length squared. The gauge fields on such a space-time are considered functions of the noncommutative coordinate operators. Through Weyl-Moyal correspondence the noncommutative algebra of operators generated by (1.1) can be represented on the algebra of ordinary functions on classical space-time by using the noncommutative Moyal $\star$-product. The gauge theories defined by the $\star$-action of the gauge algebra generators obey a no-go theorem [5], which strongly restricts the model building [10, 11].

As $\theta^{\mu\nu}$ is constant, the Lorentz invariance of (1.1) and, consequently, of the corresponding field theory built on such a space-time, breaks down. Nevertheless, noncommutative field theories in general, and gauge theories in particular formulated with Moyal $\star$-product, are invariant under the twisted Poincaré symmetry [12, 13]. In the case of twist-deformations, the generators of the twisted Hopf algebra act as usual on individual fields (see [14, 15] for a detailed account of the action of twisted Poincaré algebra and the meaning of twisted Poincaré invariance), which leads to the natural assumption that the no-go theorem for noncommutative gauge theories could be circumvented if the action of the gauge generators were to be expressed by a twist. However, it was proven [9] that the concept of twist symmetry, originally obtained for the noncommutative space-time (1.1), cannot be extended to include internal gauge symmetry. In other words, it is not possible to obtain a gauge covariant twist if the property (1.1) is adopted with $\theta^{\mu\nu}$ constant.

The same result appears to be valid also in the case of noncommutative gauge theory of gravitation. In ref. [16] it has been shown that the twisted Poincaré symmetry cannot

JHEP07(2009)084

be gauged by generalizing the Abelian twist to a covariant non-Abelian twist, nor by introducing a more general covariant twist element defined with $\theta^{\mu\nu}$ constant. Other methods used to formulate a noncommutative theory of gravitation [17–22] suffer from the same difficulty or from the restrictions of the no-go theorem.

As the introduction of a gauge covariant twist, defined with $\theta^{\mu\nu}$ constant, breaks the associativity of the algebra of functions on noncommutative space-time, both in the internal and external gauge symmetry cases, we may have to consider space-time geometries that are also non-associative, not only noncommutative. Indeed, there exist in the literature works on constructing non-associative theories with some desired properties (see, e.g., [23–27] and references therein). However, non-associativity introduces many difficulties in formulating gauge models and they are practically non-attractive.

One possible way to define a covariant star-product satisfying the associativity property is to consider models of noncommutativity with $\theta^{\mu\nu}$ depending on coordinates. In ref. [28] a new covariant star-product between differential forms has been defined. For ordinary functions, which are differential forms of order zero, this product reduces to that given by Kontsevich [29] (see also [30] for results up to the fourth order in θ and [31] for a path integral approach). The property of associativity of the new covariant star-product has been explicitly verified up to the second order in the noncommutativity parameter in ref. [28].

In this paper we extend the definition given in ref. [28] to the case of Lie algebra valued differential forms, with the ultimate aim of constructing noncommutative gauge field theories. Thus we obtain a graded Lie algebra valued Poisson algebra where the star-bracket operation can be both commutator and anti-commutator, depending on the grades of the two forms and the order in θ of the considered term in the star-product. The space-time is supposed to be a symplectic manifold on which a Poisson bracket is defined.

In section 2 we give the definition of the star-product between two arbitrary Lie algebra valued differential forms and some of their properties. Then, the star-bracket between such differential forms is introduced and some examples are given. Section 3 is devoted to the noncommutative gauge theory formulated with the new gauge covariant star-product. The noncommutative Lie algebra valued gauge potential and the field strength two-form are defined and their gauge transformation laws are established. It is shown that the field strength is gauge covariant and satisfies a deformed Bianchi identity. In section 4 a gauge covariant noncommutative metric on the space-time manifold is introduced and the action for the gauge fields is written using the star-product. It is proven that this action is gauge invariant to the second order in θ. Section 5 is dedicated to the discussion of the results and to an interpretation of noncommutative gauge theory formulated by using the star-product between Lie algebra valued differential forms on symplectic manifolds.

2 Definition of star-product

We consider a noncommutative space-time M endowed with the coordinates x^μ, $\mu = 0, 1, 2, 3$, satisfying the star-commutation relation

$$[x^\mu, x^\nu]_\star = i\theta^{\mu\nu}(x), \tag{2.1}$$

JHEP07(2009)084

where $\theta^{\mu\nu}(x) = -\theta^{\nu\mu}(x)$ is a Poisson bivector [28] satisfying

$$\theta^{\mu\rho}\partial_\rho\theta^{\nu\sigma} + \theta^{\nu\rho}\partial_\rho\theta^{\sigma\mu} + \theta^{\sigma\rho}\partial_\rho\theta^{\mu\nu} = 0\,. \tag{2.2}$$

This Poisson bivector defines a Poisson bracket between two functions $f(x)$ and $g(x)$ by

$$\{f, g\} = \theta^{\mu\nu}\,\partial_\mu f\,\partial_\nu g\,. \tag{2.3}$$

(The condition (2.2) ensures the validity of the Jacobi identity for the defined Poisson bracket.)

If a Poisson bracket is defined on M, then M is called a Poisson manifold (see [32] for mathematical details). Suppose now that the bivector $\theta^{\mu\nu}(x)$ has an inverse $\omega_{\mu\nu}(x)$, i.e.

$$\theta^{\mu\rho}\omega_{\rho\nu} = \delta^\mu_\nu\,. \tag{2.4}$$

If $\omega = \frac{1}{2}\omega_{\mu\nu}dx^\mu \wedge dx^\nu$ is nondegenerate ($\det\omega_{\mu\nu} \neq 0$) and closed ($d\omega = 0$), then it is called a symplectic two-form and M — a symplectic manifold. From now on we denote the exterior product of two forms α and β simply by $\alpha\beta$ and understand that it means $\alpha\wedge\beta$. It can be verified that the condition $d\omega = 0$ is equivalent to the equation (2.2) [28, 33]. In this paper we shall consider only the case when M is symplectic.

Because the gauge theories involve Lie algebra valued differential forms such as $A = A^a_\mu\,T_a\,dx^\mu = A_\mu\,dx^\mu$, $A_\mu = A^a_\mu(x)\,T_a$, where T_a are the infinitesimal generators of a symmetry Lie group G, we need to generalize the definition of the Poisson bracket to differential forms and define then an associative star-product for such cases. These issues were solved in ref. [28, 33, 34] and here we just recall the definitions and properties to fix the idea. However, we generalize these results to the case of Lie algebra valued forms. This means that the Poisson algebra becomes a graded Lie algebra valued one. Therefore, the commutator of differential forms can be a commutator or an anti-commutator, depending on their degrees.

Assuming that $\theta^{\mu\nu}(x)$ is invertible, we can always write the Poisson bracket $\{x, dx\}$ in the form [28, 33]

$$\{x^\mu, dx^\nu\} = -\theta^{\mu\sigma}\Gamma^\nu_{\sigma\rho}dx^\rho\,, \tag{2.5}$$

where $\Gamma^\nu_{\sigma\rho}$ are some functions of x transforming like a connection under general coordinate transformations. As $\Gamma^\nu_{\sigma\rho}$ is generally not symmetric, on can use the connection one-forms

$$\widetilde{\Gamma}^\mu_\nu = \Gamma^\mu_{\nu\rho}dx^\rho, \qquad \Gamma^\mu_\nu = dx^\rho\Gamma^\mu_{\rho\nu} \tag{2.6}$$

to define two kinds of covariant derivatives, $\widetilde{\nabla}$ and ∇, respectively. For example, if $\alpha = \alpha_\nu\,dx^\nu$ is a one-form, then

$$\widetilde{\nabla}_\mu\alpha = (\partial_\mu\alpha_\nu - \Gamma^\rho_{\mu\nu}\alpha_\rho)\,dx^\nu \tag{2.7}$$

and analogously for $\nabla_\mu\alpha$. Given θ and Γ, all Poisson brackets are determined [33]. Now, if α and β are two arbitrary differential forms, then their Poisson bracket is given by [28]

$$\{\alpha, \beta\} = \theta^{\mu\nu}\nabla_\mu\alpha\nabla_\nu\beta + (-1)^{|\alpha|}\widetilde{R}^{\mu\nu}(i_\mu\alpha)(i_\nu\beta)\,, \tag{2.8}$$

where

$$\widetilde{R}^{\mu\nu} = \frac{1}{2}\widetilde{R}^{\mu\nu}_{\rho\sigma}\, dx^\rho dx^\sigma \tag{2.9}$$

and

$$\widetilde{R}^{\nu}_{\lambda\rho\sigma} = \partial_\rho \Gamma^\nu_{\lambda\sigma} - \partial_\sigma \Gamma^\nu_{\lambda\rho} + \Gamma^\nu_{\tau\rho}\Gamma^\tau_{\lambda\sigma} - \Gamma^\nu_{\tau\sigma}\Gamma^\tau_{\lambda\rho}\,, \tag{2.10}$$

while i_μ denotes the interior product which maps k-forms into $(k-1)$-forms. More exactly, if α is the k-form

$$\alpha = \frac{1}{k!}\alpha_{\mu_1\ldots\mu_k} dx^{\mu_1}\ldots dx^{\mu_k}\,, \tag{2.11}$$

then

$$i_\mu \alpha = \frac{1}{(k-1)!}\alpha_{\mu\mu_2\ldots\mu_k} dx^{\mu_2}\ldots dx^{\mu_k}\,. \tag{2.12}$$

In order that (2.8) satisfies the properties of the Poisson bracket, the following conditions must be imposed [28]:

a) $\theta^{\mu\nu}$ satisfies the Jacobi identity (2.2);

b) $\widetilde{\nabla}_\rho$ is symplectic, i.e. $\widetilde{\nabla}_\rho \theta^{\mu\nu} = 0$;

c) the connection ∇_ν has vanishing curvature, i.e. $[\nabla_\mu, \nabla_\nu]\alpha = 0$ for any differential form α;

d) the curvature $\widetilde{R}^{\mu\nu}$ associated to the connection one-form $\widetilde{\Gamma}^\rho_\sigma$ is covariantly constant under ∇_ρ, i.e. $\nabla_\rho \widetilde{R}^{\mu\nu} = 0$. The curvature $\widetilde{R}^{\mu\nu}$ is defined as in the eqs. (2.9)–(2.10).

Using the above properties, a star-product between differential forms has been defined in ref. [28]. Here we extend this definition to the case of Lie algebra valued differential forms. If $\alpha = \alpha^a\, T_a$ and $\beta = \beta^b\, T_b$ are two arbitrary such forms, where α^a and β^b are ordinary differential forms of degrees $|\alpha|$ and respectively $|\beta|$, and T_a are Lie algebra generators in the fundamental representation, then their star-product has the expression

$$\alpha \star \beta = \alpha\beta + \sum_{n=1}^{\infty}\left(\frac{i\hbar}{2}\right)^n C_n(\alpha,\beta) = \alpha^a\beta^b\, T_a T_b + \sum_{n=1}^{\infty}\left(\frac{i\hbar}{2}\right)^n C_n(\alpha^a,\beta^b)\, T_a T_b\,, \tag{2.13}$$

where $C_n(\alpha^a, \beta^b)$ are bilinear differential operators. We impose then the condition that the star-product (2.13) satisfies the property of associativity

$$(\alpha \star \beta)\star\gamma = \alpha\star(\beta\star\gamma)\,. \tag{2.14}$$

Introducing (2.13) in (2.14), we find the following general condition of associativity for an arbitrary order n:

$$\begin{aligned} &C_n(C_0(\alpha,\beta),\gamma) + C_{n-1}(C_1(\alpha,\beta),\gamma) + C_{n-2}(C_2(\alpha,\beta),\gamma)\ldots + C_0(C_n(\alpha,\beta),\gamma) \\ &\quad = C_n(\alpha, C_0(\beta,\gamma)) + C_{n-1}(\alpha, C_1(\beta,\gamma)) + C_{n-2}(\alpha, C_2(\beta,\gamma))\ldots + C_0(\alpha, C_n(\beta,\gamma))\,. \end{aligned} \tag{2.15}$$

JHEP07(2009)084

In ref. [28] the expressions of the operators $C_n(\alpha^a, \beta^b)$ were obtained up to the second order in θ. We admit that these results are also valid in our case of Lie algebra valued differential forms with adequate definitions. They are

$$C_1(\alpha^a,\beta^b) \equiv \{\alpha^a,\beta^b\} = \theta^{\mu\nu}\left[\nabla_\mu\alpha^a\nabla_\nu\beta^b + (-1)^{|\alpha|}\widetilde{R}^{\rho\sigma}_{\mu\nu}(i_\rho\alpha^a)(i_\sigma\beta^b)\right], \tag{2.16}$$

$$\begin{aligned} C_2(\alpha^a,\beta^b) = &\frac{1}{2}\theta^{\mu\nu}\theta^{\rho\sigma}\nabla_\mu\nabla_\rho\alpha^a\nabla_\nu\nabla_\sigma\beta^b + \frac{1}{3}\theta^{\mu\rho}\partial_\rho\theta^{\nu\sigma}(\nabla_\mu\nabla_\nu\alpha^a\nabla_\sigma\beta^b - \nabla_\nu\alpha^a\nabla_\mu\nabla_\sigma\beta^b) \\ &-\frac{1}{2}\widetilde{R}^{\mu\nu}\widetilde{R}^{\rho\sigma}(i_\mu i_\rho\alpha^a)(i_\nu i_\sigma\beta^b) \\ &-\frac{1}{3}\widetilde{R}^{\mu\nu}(i_\nu\widetilde{R}^{\rho\sigma})[(-1)^{|\alpha|}(i_\mu i_\rho\alpha^a)(i_\sigma\beta^b) + (i_\rho\alpha^a)(i_\mu i_\sigma\beta^b)] \\ &+(-1)^{|\alpha|}\theta^{\mu\nu}\widetilde{R}^{\rho\sigma}(i_\rho\nabla_\mu\alpha^a)(i_\sigma\nabla_\nu\beta^b). \end{aligned} \tag{2.17}$$

It is important to observe that the operators $C_n(\alpha^a, \beta^b)$ have the generalized Moyal symmetry [28],

$$C_n(\alpha^a,\beta^b) = (-1)^{|\alpha||\beta|+n}C_n(\beta^b,\alpha^a). \tag{2.18}$$

Taking into account the graded structure of our Poisson algebra, we define the star commutator of two Lie algebra valued differential forms $\alpha = \alpha^a\, T_a$ and $\beta = \beta^b\, T_b$ by

$$[\alpha,\beta]_\star = \alpha\star\beta - (-1)^{|\alpha||\beta|}\beta\star\alpha. \tag{2.19}$$

For example, if α and β are Lie algebra valued one-forms, we have

$$[\alpha,\beta]_\star = \alpha^a\beta^b[T_a,T_b] + \frac{i\hbar}{2}C_1(\alpha^a,\beta^b)\{T_a,T_b\} + \left(\frac{i\hbar}{2}\right)^2 C_2(\alpha^a,\beta^b)[T_a,T_b] + \ldots \tag{2.20}$$

This result shows that the star commutator of Lie algebra valued differential forms does not close in general in the Lie algebra but in its universal enveloping algebra. Only in the case of the unitary groups $\mathrm{U}(n)$ the universal enveloping algebra $\mathcal{U}(u(n))$ in the fundamental representation coincides with the Lie algebra $u(n)$ (of $n\times n$ antihermitian matrices).

We shall use all these properties in sections 3 and 4 to develop a noncommutative gauge theory with non-Abelian gauge group, up to the second order in $\hbar$ (or equivalently $\mathcal{O}(\theta^3)$).

3 Noncommutative gauge theory

Consider the gauge group G whose infinitesimal generators satisfy the algebra

$$[T_a,T_b] = if^c_{ab}T_c, \qquad a,b,c = 1,2,\ldots,m, \tag{3.1}$$

with the structure constants $f^a_{bc} = -f^a_{cb}$ and the Lie algebra valued infinitesimal parameter

$$\hat{\lambda} = \hat{\lambda}^a T_a. \tag{3.2}$$

We use the hat symbol to denote the noncommutative quantities of our gauge theory. The parameter $\hat{\lambda}$ is a 0-form, i.e. $\hat{\lambda}^a$ are functions of the coordinates x^μ on the symplectic manifold M.

Now we define the gauge transformation of the noncommutative Lie algebra valued gauge potential

$$\hat{A} = \hat{A}^a_\mu(x)\, T_a\, dx^\mu = \hat{A}_\mu\, dx^\mu, \quad \hat{A}_\mu = \hat{A}^a_\mu(x)\, T_a\,, \tag{3.3}$$

by

$$\hat{\delta}\hat{A} = d\hat{\lambda} - i[\hat{A}, \hat{\lambda}]_\star\,. \tag{3.4}$$

Here we consider the formula (2.19) for the commutator $[\hat{A}, \hat{\lambda}]_\star$. Then, using the definition (2.13) of the star-product, we can write (3.4) as

$$\hat{\delta}\hat{A}^a = d\hat{\lambda}^a + f^a_{bc}\hat{A}^b\hat{\lambda}^c + \frac{\hbar}{2} d^a_{bc} C_1(\hat{A}^b, \hat{\lambda}^c) - \frac{\hbar^2}{4} f^a_{bc} C_2(\hat{A}^b, \hat{\lambda}^c) + \dots\,, \tag{3.5}$$

where we denoted $\{T_a, T_b\} = d^c_{ab} T_c$. Since $\hat{\lambda}^a$ are functions, the operators $C_1(\hat{A}^b, \hat{\lambda}^c)$ and $C_2(\hat{A}^b, \hat{\lambda}^c)$ have the expressions (see eqs. (2.16)-(2.17))

$$\begin{aligned} C_1(\hat{A}^b, \hat{\lambda}^c) &\equiv \{\hat{A}^b, \hat{\lambda}^c\} = \theta^{\mu\nu}\nabla_\mu \hat{A}^b \partial_\nu \hat{\lambda}^c\,, \qquad (3.6)\\ C_2(\hat{A}^b, \hat{\lambda}^c) &= \frac{1}{2}\theta^{\mu\nu}\theta^{\rho\sigma}\nabla_\mu\nabla_\rho \hat{A}^b \partial_\nu\partial_\sigma \hat{\lambda}^c + \frac{1}{3}\theta^{\mu\nu}\partial_\nu\theta^{\rho\sigma}(\nabla_\mu\nabla_\rho \hat{A}^b\partial_\sigma\hat{\lambda}^c - \nabla_\rho\hat{A}^b\partial_\mu\partial_\sigma\hat{\lambda}^c)\,. \end{aligned}$$

Here we use the definition of the covariant derivative

$$\nabla_\mu \hat{A}^a = (\partial_\mu \hat{A}^a_\nu - \Gamma^\rho_{\mu\nu}\hat{A}^a_\rho)dx^\nu \equiv (\nabla_\mu \hat{A}^a_\nu)dx^\nu\,. \tag{3.7}$$

In particular, in the case when the gauge group G is U(1), we have $f^a_{bc} = 0$, $a, b, c = 1$, $d^1_{11} = 2$, $\hat{A}^1 \equiv \hat{A}$, $\hat{\lambda}^1 \equiv \hat{\lambda}$, therefore (3.5) becomes

$$\hat{\delta}\hat{A} = d\hat{\lambda} + \hbar C_1(\hat{A}, \hat{\lambda}) + \mathcal{O}(\hbar^3)\,. \tag{3.8}$$

More explicitly, in terms of components,

$$\hat{\delta}\hat{A}_\mu = \partial_\mu\hat{\lambda} + \hbar\theta^{\nu\sigma}\nabla_\nu\hat{A}_\mu\partial_\sigma\hat{\lambda} + \mathcal{O}(\hbar^3)\,, \tag{3.9}$$

where $\hat{A}_\mu(x)$ is the U(1) gauge potential and $\hat{\lambda}(x)$ — the infinitesimal parameter (phase). In zeroth approximation (3.9) reproduces the usual U(1) gauge potential transformation.

Analogously, in the case of U(2), we consider $a = (0, k)$, $k = 1, 2, 3$. Then, $f^k_{ij} = 2\epsilon_{ijk}$, $\{T_i, T_j\} = 2\delta_{ij}T_0$, $i, j, k = 1, 2, 3$ and $\{T_0, T_k\} = \{T_k, T_0\} = 2T_k$, $\{T_0, T_0\} = 2T_0$, where $T_0 = I$ is the unit matrix and $T_k = \sigma_k$ — the Pauli matrices as generators of SU(2). Then, since

$$\hat{A} = \hat{A}^k\sigma^k + \hat{A}^0 I, \quad \hat{\lambda} = \hat{\lambda}^k\sigma_k + \hat{\lambda}^0 I\,, \tag{3.10}$$

we obtain from (3.5)

$$\hat{\delta}\hat{A}^0 = d\hat{\lambda}^0 + \hbar\left[C_1(\hat{A}^i, \hat{\lambda}^j)\delta_{ij} + C_1(\hat{A}^0, \hat{\lambda}^0)\right] + \mathcal{O}(\hbar^3)\,, \tag{3.11}$$

$$\hat{\delta}\hat{A}^k = d\hat{\lambda}^k + 2\epsilon_{ijk}\hat{A}^i\hat{\lambda}^j + \hbar\left[C_1(\hat{A}^k, \hat{\lambda}^0) + C_1(\hat{A}^0, \hat{\lambda}^k)\right] - \frac{\hbar^2}{2}C_2(\hat{A}^i, \hat{A}^j)\epsilon_{ijk} + \mathcal{O}(\hbar^3). \tag{3.12}$$

Here, the quantities $(\hat{A}^0, \hat{\lambda}^0)$ correspond to the noncommutative U(1) sector and $(\hat{A}^k, \hat{\lambda}^k)$, $k = 1, 2, 3$ — to the noncommutative SU(2) sector. Considering, in addition, the restriction

JHEP07(2009)084

U(2) $\to$ U(1), we must keep only eq. (3.11), but without the term $C_1(\hat{A}^i, \hat{\lambda}^j)\delta_{ij}$, $i, j = 1, 2, 3$. Thus, we rediscover the previous U(1) result (3.8) with $\hat{A}^0 \equiv \hat{A}$, $\hat{\lambda}^0 \equiv \hat{\lambda}$.

We define the curvature two-form $\hat{F}$ of the gauge potentials by

$$\hat{F} = \frac{1}{2} dx^\mu dx^\nu \hat{F}_{\mu\nu} = d\hat{A} - \frac{i}{2}[\hat{A}, \hat{A}]_\star . \tag{3.13}$$

Then, using the definition (2.13) of the star-product and the property (2.18) of the operators $C_n(\alpha^a, \beta^b)$, we obtain from (3.13)

$$\hat{F}^a = d\hat{A}^a + \frac{1}{2} f^a_{bc} \hat{A}^b \hat{A}^c + \frac{1}{2}\frac{\hbar}{2} d^a_{bc} C_1(\hat{A}^b, \hat{A}^c) - \frac{1}{2}\frac{\hbar^2}{4} f^a_{bc} C_2(\hat{A}^b, \hat{A}^c) + \mathcal{O}(\hbar^3) . \tag{3.14}$$

More explicitly, in terms of components we have

$$\hat{F}^a_{\mu\nu} = \partial_\mu \hat{A}^a_\nu - \partial_\nu \hat{A}^a_\mu + f^a_{bc} \hat{A}^b_\mu \hat{A}^c_\nu + \frac{\hbar}{2} d^a_{bc} C_1(\hat{A}^b_\mu, \hat{A}^c_\nu) - \frac{\hbar^2}{4} f^a_{bc} C_2(\hat{A}^b_\mu, \hat{A}^c_\nu) + \mathcal{O}(\hbar^3) , \tag{3.15}$$

where we used the definition $C_n(\hat{A}^b, \hat{A}^c) = C_n(\hat{A}^b_\mu, \hat{A}^c_\nu) dx^\mu dx^\nu$, with

$$C_1(\hat{A}^b_\mu, \hat{A}^c_\nu) = \theta^{\rho\sigma} \left[\nabla_\rho \hat{A}^b_\mu \nabla_\sigma \hat{A}^c_\nu - \widetilde{R}^\alpha_{\sigma\mu\nu} \hat{A}^b_\rho \hat{A}^c_\alpha \right] , \tag{3.16}$$

$$\begin{aligned} C_2(\hat{A}^b_\mu, \hat{A}^c_\nu) = {}& \theta^{\rho\sigma}\theta^{\lambda\tau} \left[\frac{1}{2} \nabla_\rho \nabla_\lambda \hat{A}^b_\mu \nabla_\sigma \nabla_\tau \hat{A}^c_\nu - \widetilde{R}^\alpha_{\tau\mu\nu} \nabla_\rho \hat{A}^b_\lambda \nabla_\sigma \hat{A}^c_\alpha \right] \\ & + \frac{1}{3} \theta^{\rho\sigma} \partial_\sigma \theta^{\lambda\tau} (\nabla_\rho \nabla_\lambda \hat{A}^b_\mu \nabla_\tau \hat{A}^c_\nu - \nabla_\lambda \hat{A}^b_\mu \nabla_\rho \nabla_\tau \hat{A}^c_\nu) . \end{aligned} \tag{3.17}$$

In the particular case of the U(1) gauge group we obtain

$$\hat{F}_{\mu\nu} = \partial_\mu \hat{A}_\nu - \partial_\nu \hat{A}_\mu + \hbar \theta^{\rho\sigma} \left[\nabla_\rho \hat{A}_\mu \nabla_\sigma \hat{A}_\nu - \frac{1}{2} \widetilde{R}^\alpha_{\sigma\mu\nu} \hat{A}_\rho \hat{A}_\alpha \right] + \mathcal{O}(\hbar^3) . \tag{3.18}$$

Under the gauge transformation (3.4), the curvature 2-form $\hat{F}$ transforms as

$$\hat{\delta} \hat{F} = i[\hat{\lambda}, \hat{F}]_\star , \tag{3.19}$$

where we used the Leibniz rule

$$d(\hat{\alpha} \star \hat{\beta}) = d\hat{\alpha} \star \beta + (-1)^{|\alpha|} \hat{\alpha} \star d\hat{\beta}, \tag{3.20}$$

which we admit to be valid to all orders in $\hbar$. In terms of components, (3.19) reads

$$\hat{\delta} \hat{F}^a = f^a_{bc} \hat{F}^b \hat{\lambda}^c + \frac{\hbar}{2} d^a_{bc} C_1(\hat{F}^b, \hat{\lambda}^c) - \frac{\hbar^2}{4} f^a_{bc} C_2(\hat{F}^b, \hat{\lambda}^c) + \mathcal{O}(\hbar^3) . \tag{3.21}$$

If the gauge group is U(1), then we obtain

$$\hat{\delta} \hat{F} = \hbar C_1(\hat{F}, \hat{\lambda}) = \hbar \theta^{\rho\sigma} \nabla_\rho \hat{F} \partial_\sigma \hat{\lambda} + \mathcal{O}(\hbar^3) ,$$

or, in terms of components,

$$\hat{\delta} \hat{F}_{\mu\nu} = \hbar \theta^{\rho\sigma} \nabla_\rho \hat{F}_{\mu\nu} \partial_\sigma \hat{\lambda} + \mathcal{O}(\hbar^3) . \tag{3.22}$$

JHEP07(2009)084

Also, in the zeroth order, the formula (3.21) becomes

$$\delta F^a_{\mu\nu} = f^a_{bc} F^b_{\mu\nu} \lambda^c \Longleftrightarrow \delta F = i[\lambda, F] \,. \tag{3.23}$$

This formula reproduces therefore the result of the commutative gauge theory. Using again the Leibniz rule, we obtain the deformed Bianchi identity

$$d\hat{F} - i[\hat{A}, \hat{F}]_\star = 0 \,. \tag{3.24}$$

If we apply the definition (2.19) of the star commutator, we obtain

$$d\hat{F} + i[\hat{F}, \hat{A}] = \left[\frac{\hbar}{2} d^a_{bc} C_1(\hat{F}^b, \hat{A}^c) - \frac{\hbar^2}{4} f^a_{bc} C_2(\hat{F}^b, \hat{A}^c)\right] T_a + \mathcal{O}(\hbar^3) \,, \tag{3.25}$$

or, in terms of components,

$$d\hat{F}^a - f^a_{bc} \, \hat{F}^b \hat{A}^c = \frac{\hbar}{2} d^a_{bc} C_1(\hat{F}^b, \hat{A}^c) - \frac{\hbar^2}{4} f^a_{bc} C_2(\hat{F}^b, \hat{A}^c) + \mathcal{O}(\hbar^3) \,. \tag{3.26}$$

We remark that in the zeroth order we obtain from (3.25) the usual Bianchi identity

$$dF - i[A, F] = 0 \,. \tag{3.27}$$

In addition, if the gauge group is U(1), the Bianchi identity (3.26) becomes

$$d\hat{F} = \hbar \, C_1(\hat{F}, \hat{A}) + \mathcal{O}(\hbar^3) \,. \tag{3.28}$$

This result is also in accord with that of ref. [33].

4 Noncommutative Yang-Mills action

Having established the previous results, we can construct a noncommutative Yang-Mills action. Denote the metric in the noncommutative space-time M by $G^{\nu\rho}$. Its covariant derivative is

$$\nabla_\mu G^{\nu\rho} = \partial_\mu G^{\nu\rho} + G^{\nu\sigma} \Gamma^\rho_{\mu\sigma} + \Gamma^\nu_{\mu\sigma} G^{\sigma\rho} \,. \tag{4.1}$$

If $G^{\nu\rho}$ is not constant, we have to modify it to be a gauge covariant metric $\hat{G}^{\nu\rho}$ for the noncommutative Yang-Mills action. The metric $\hat{G}^{\nu\rho}$ is gauge covariant in the sense that it transforms like $\hat{F}$ (see (3.19))

$$\hat{\delta} \hat{G}^{\mu\nu} = i[\hat{\lambda}, \hat{G}^{\mu\nu}]_\star \,. \tag{4.2}$$

Then, using the definition (2.19) of the star commutator, we obtain from (4.2)

$$\hat{\delta} \hat{G}^{\mu\nu} = \hbar \theta^{\rho\sigma} \nabla_\rho \hat{G}^{\mu\nu} \partial_\sigma \hat{\lambda} + \mathcal{O}(\hbar^3) \,. \tag{4.3}$$

The explicit form of the metric could be obtained using, for example, the Seiberg-Witten map [1] extended to the new star-product defined in ref. [28] and used by us in developing the noncommutative gauge theory.

JHEP07(2009)084

Define the noncommutative Yang-Mills action by (see ref. [33])

$$\hat{S}_{\rm NC} = -\frac{1}{2g^2}\langle Tr(\hat{G}\star\hat{F}\star\hat{G}\star\hat{F})\rangle = -\frac{1}{4g^2}\langle \hat{G}^{\mu\rho}\star\hat{F}_{\rho\nu}\star\hat{G}^{\nu\sigma}\star\hat{F}_{\sigma\mu}\rangle\,, \tag{4.4}$$

where g is the gauge coupling constant, and we have used the normalization property

$$Tr(T_aT_b) = \frac{1}{2}\delta_{ab}I\,. \tag{4.5}$$

Using the properties of gauge covariance (3.19) and (4.2) for $\hat{F}$ and $\hat{G}$ respectively, we obtain

$$\hat{\delta}\hat{S}_{\rm NC} = -\frac{1}{4g^2}\hbar\,\langle C_1(Tr(\hat{G}\hat{F}\hat{G}\hat{F}),\hat{\lambda})\rangle + \mathcal{O}(\hbar^3)\,. \tag{4.6}$$

Now, since the integral is cyclic in the Poisson limit [33], i.e.

$$\langle C_1(Tr(\hat{G}\hat{F}\hat{G}\hat{F}),\hat{\lambda})\rangle = 0\,, \tag{4.7}$$

eq. (4.6) becomes

$$\hat{\delta}\hat{S}_{\rm NC} = 0\,. \tag{4.8}$$

Therefore, the action $\hat{\delta}\hat{S}_{\rm NC}$ is invariant up to the second order in θ (or $\hbar$). In ref. [33] it has been proven that the action (4.4) can be further simplified as

$$\hat{S}_{\rm NC} = -\frac{1}{2g^2}\langle Tr(\hat{G}\hat{F}\hat{G}\hat{F})\rangle + \mathcal{O}(\hbar^3)\,. \tag{4.9}$$

Imposing then the variational principle $\hat{\delta}_{\hat{A}}\hat{S}_{\rm NC} = 0$ with respect to the noncommutative gauge fields $\hat{A}^a_\mu$, we can obtain the noncommutative Yang-Mills field equations.

5 Discussion

We have developed a noncommutative gauge theory by using a star-product between differential forms on symplectic manifolds defined as in ref. [28], by extending the definition given in ref. [28] to the case of Lie algebra valued differential forms. In this manner we have obtained a graded Lie algebra valued Poisson algebra where the star-bracket operation can be both commutator and anti-commutator, depending on the grades of the two forms. The graded, but not Lie algebra valued, Poisson algebra on a symplectic manifold was initially introduced in refs. [33] and [34]. In these papers an explicit form of the bracket for one-forms has been obtained. In ref. [28] the results have been generalized to the case of graded Poisson bracket for arbitrary degrees of differential forms. In order to develop a noncommutative gauge theory we have defined the star commutator of two Lie algebra valued differential forms. Since the star-product does not close in general in the Lie algebra, but only in its universal enveloping algebra, we can use the unitary Lie algebras U(n) as gauge symmetry or extend the results to the Hopf algebra for any other algebras. We have introduced the noncommutative one-form gauge potentials $\hat{A}$ and the field strength two-form $\hat{F}$ and have obtained their gauge transformation laws. We have proven that the defined field strength $\hat{F}$ is gauge covariant and satisfies a deformed Bianchi

JHEP07(2009)084

identity. To obtain these results we have used the Leibniz rule which we admit to be valid to all orders in θ.

Finally, we have defined an action for the gauge fields by introducing a gauge covariant noncommutative metric $\hat{G}^{\nu\rho}$ on the space-time manifold. The gauge invariance of this action has been verified up to the second order in θ using the property that the integral is cyclic in the Poisson limit [33]. The explicit form of the metric $\hat{G}^{\nu\rho}$ could be obtained using, for instance, the Seiberg-Witten map extended to the new star-product. Extending the gauge theory to higher orders in θ requires to find the explicit expressions for the bilinear differential operators $C_n(\alpha^a, \beta^b)$ which define the star-product.

Acknowledgments

We would like to thank Markku Oksanen for useful comments. The support of the Academy of Finland under the Projects No. 121720 and 127626 is greatly acknowledged. G. Z. acknowledges the support of CNCSIS-UEFISCSU Grant No. 620 of the Ministry of Education and Research of Romania.

References

[1] N. Seiberg and E. Witten, *String theory and noncommutative geometry*, *JHEP* **09** (1999) 032 [hep-th/9908142] [SPIRES].

[2] J.M. Gracia-Bondia and C.P. Martin, *Chiral gauge anomalies on noncommutative R^4*, *Phys. Lett.* **B 479** (2000) 321 [hep-th/0002171] [SPIRES].

[3] J. Madore, S. Schraml, P. Schupp and J. Wess, *Gauge theory on noncommutative spaces*, *Eur. Phys. J.* **C 16** (2000) 161 [hep-th/0001203] [SPIRES].

[4] B. Jurčo, S. Schraml, P. Schupp and J. Wess, *Enveloping algebra valued gauge transformations for non-abelian gauge groups on non-commutative spaces*, *Eur. Phys. J.* **C 17** (2000) 521 [hep-th/0006246] [SPIRES].

[5] M. Chaichian, P. Prešnajder, M.M. Sheikh-Jabbari and A. Tureanu, *Noncommutative gauge field theories: a no-go theorem*, *Phys. Lett.* **B 526** (2002) 132 [hep-th/0107037] [SPIRES].

[6] W. Behr and A. Sykora, *Construction of gauge theories on curved noncommutative spacetime*, *Nucl. Phys.* **B 698** (2004) 473 [hep-th/0309145] [SPIRES].

[7] P. Aschieri, M. Dimitrijević, F. Meyer, S. Schraml and J. Wess, *Twisted gauge theories*, *Lett. Math. Phys.* **78** (2006) 61 [hep-th/0603024] [SPIRES].

[8] M. Chaichian and A. Tureanu, *Twist symmetry and gauge invariance*, *Phys. Lett.* **B 637** (2006) 199 [hep-th/0604025] [SPIRES].

[9] M. Chaichian, A. Tureanu and G. Zet, *Twist as a symmetry principle and the noncommutative gauge theory formulation*, *Phys. Lett.* **B 651** (2007) 319 [hep-th/0607179] [SPIRES].

[10] M. Chaichian, P. Prešnajder, M.M. Sheikh-Jabbari and A. Tureanu, *Noncommutative standard model: model building*, *Eur. Phys. J.* **C 29** (2003) 413 [hep-th/0107055] [SPIRES];

M. Chaichian, A. Kobakhidze and A. Tureanu, *Spontaneous reduction of noncommutative gauge symmetry and model building*, Eur. Phys. J. **C 47** (2006) 241 [hep-th/0408065] [SPIRES].

[11] X. Calmet, B. Jurčo, P. Schupp, J. Wess and M. Wohlgenannt, *The standard model on non-commutative space-time*, Eur. Phys. J. **C 23** (2002) 363 [hep-ph/0111115] [SPIRES].

[12] M. Chaichian, P.P. Kulish, K. Nishijima and A. Tureanu, *On a Lorentz-invariant interpretation of noncommutative space-time and its implications on noncommutative QFT*, Phys. Lett. **B 604** (2004) 98 [hep-th/0408069] [SPIRES].

[13] M. Chaichian, P. Prešnajder and A. Tureanu, *New concept of relativistic invariance in NC space-time: twisted Poincaré symmetry and its implications*, Phys. Rev. Lett. **94** (2005) 151602 [hep-th/0409096] [SPIRES].

[14] M. Chaichian, P.P. Kulish, A. Tureanu, R.B. Zhang and X. Zhang, *Noncommutative fields and actions of twisted Poincaré algebra*, J. Math. Phys. **49** (2008) 042302 [arXiv:0711.0371] [SPIRES].

[15] M. Chaichian, K. Nishijima, T. Salminen and A. Tureanu, *Noncommutative quantum field theory: a confrontation of symmetries*, JHEP **06** (2008) 078 [arXiv:0805.3500] [SPIRES].

[16] M. Chaichian, M. Oksanen, A. Tureanu and G. Zet, *Gauging the twisted Poincaré symmetry as noncommutative theory of gravitation*, Phys. Rev. **D 79** (2009) 044016 [arXiv:0807.0733] [SPIRES].

[17] P. Aschieri et al., *A gravity theory on noncommutative spaces*, Class. Quant. Grav. **22** (2005) 3511 [hep-th/0504183] [SPIRES].

[18] L. Álvarez-Gaumé, F. Meyer and M.A. Vazquez-Mozo, *Comments on noncommutative gravity*, Nucl. Phys. **B 753** (2006) 92 [hep-th/0605113] [SPIRES].

[19] E. Harikumar and V.O. Rivelles, *Noncommutative gravity*, Class. Quant. Grav. **23** (2006) 7551 [hep-th/0607115] [SPIRES].

[20] O. Bertolami and L. Guisado, *Noncommutative scalar field coupled to gravity*, Phys. Rev. **D 67** (2003) 025001 [gr-qc/0207124] [SPIRES].

[21] A.H. Chamseddine, *Deforming Einstein's gravity*, Phys. Lett. **B 504** (2001) 33 [hep-th/0009153] [SPIRES].

[22] A.H. Chamseddine, *Invariant actions for noncommutative gravity*, J. Math. Phys. **44** (2003) 2534 [hep-th/0202137] [SPIRES].

[23] L. Cornalba and R. Schiappa, *Nonassociative star product deformations for D-brane worldvolumes in curved backgrounds*, Commun. Math. Phys. **225** (2002) 33 [hep-th/0101219] [SPIRES].

[24] A.I. Nesterov and L.V. Sabinin, *Nonassociative geometry: Friedmann-Robertson-Walker spacetime*, Int. J. Geom. Meth. Mod. Phys. **3** (2006) 1481 [hep-th/0406229] [SPIRES].

[25] S. Majid, *Gauge theory on nonassociative spaces*, J. Math. Phys. **46** (2005) 103519 [math.QA/0506453].

[26] Y. Sasai and N. Sasakura, *One-loop unitarity of scalar field theories on Poincaré invariant commutative nonassociative spacetimes*, JHEP **09** (2006) 046 [hep-th/0604194] [SPIRES].

[27] Y. Matsuo, *Projection operators and D-branes in purely cubic open string field theory*, Mod. Phys. Lett. **A 16** (2001) 1811 [hep-th/0107007] [SPIRES].

JHEP07(2009)084

[28] S. McCurdy, A. Tagliaferro and B. Zumino, *The star product for differential forms on symplectic manifolds*, arXiv:0809.4717 [SPIRES].

[29] M. Kontsevich, *Deformation quantization of Poisson manifolds, I*, *Lett. Math. Phys.* **66** (2003) 157 [q-alg/9709040] [SPIRES].

[30] V.G. Kupriyanov and D.V. Vassilevich, *Star products made (somewhat) easier*, *Eur. Phys. J.* **C 58** (2008) 627 [arXiv:0806.4615] [SPIRES].

[31] A.S. Cattaneo and G. Felder, *A path integral approach to the Kontsevich quantization formula*, *Commun. Math. Phys.* **212** (2000) 591 [math.QA/9902090].

[32] I. Vaisman, *Lectures on the geometry of Poisson manifolds*, Progress in Mathematics, volume 118, Birkhäuser Verlag, Berlin Germany (1994).

[33] P.-M. Ho and S.-P. Miao, *Noncommutative differential calculus for D-brane in non-constant B field background*, *Phys. Rev.* **D 64** (2001) 126002 [hep-th/0105191] [SPIRES].

[34] C.-S. Chu and P.-M. Ho, *Poisson algebra of differential forms*, *Int. J. Mod. Phys.* **12** (1997) 5573 [q-alg/9612031] [SPIRES].

JHEP07(2009)084